REPRESENTING 3-MANIFOLDS BY DEHN SPHERES

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Abstract

Let M be a closed orientable 3-manifold. A *Dehn-sphere* S is a 2-sphere immersed in M with only double curve and triple point singularities. S *fills* M if S defines a cell decomposition of M. It is proven that *every closed orientable 3-manifold has a filling Dehn-sphere.* Examples are given, and Johansson diagrams are proposed as a method for representing all closed orientable 3-manifolds.

Let M be a closed orientable 3-manifold. A *Dehn-sphere* S is a 2-sphere immersed in M with only double curve and triple point singularities.I will say that S *fills* M if S defines a cell decomposition of M. In $[H1]$ it is stated that *every homotopy 3-sphere has a filling Dehn-sphere.* Haken says that *this can be done by considering the step-by-step deformation of a 2-sphere in (the homotopy 3-sphere) M as carried out in* $[H2]$ .In $[FR]$ something weaker is stated for every M, namely *that M-S is a disjoint union of open 3-balls.* Perhaps the argument presented in $[FR]$ gives the stronger result, but I cannot follow the finer details. Moreover $[H2]$ is very hard to read and I suspect the argument relies heavily in M being a homotopy sphere. Therefore I tried several methods and I found an almost trivial proof which shows that the result is true in general, not only for homotopy 3-spheres. Since my method is constructive and produces the Dehn-sphere out of a Heegaard diagram, I decided, prompted by F . González Acuña, to publish it. I am very grateful to him for several illuminating conversations.

Theorem 1. *Every closed orientable 3-manifold has a filling Dehn-sphere.*

Proof. Start with a Heegaard diagram (F, v, w) for M. Here F is a closed orientable 2-manifold of some genus g embedded in M, which separates M into two handlebodies V and W, and v and w are, respectively, complete systems of meridians of V and W.

Each component $N(v_i)$ of a regular neighborhood of v in V can be identified with $D \times [-1,1]$ so that v_i is identified with $D \times \{0\}$. Here D is the set of complex numbers z with $\mid z \mid\leqslant 1$. Consider the 2-sphere $\Sigma = \{(z,t) \in D \times [-1,1] : t = \pm(1- \mid z \mid)\}$. Via the above identification Σ defines a 2-sphere σ_i inside $N(v_i)$ such that $\partial V \cap \ \sigma_i = \partial v_i$. Then $\partial V \cup_{i=1}^{g} \ \sigma_i$ is a Dehn-sphere in M which we will denote by $S(F,v)$, since, up to isotopy, it is determined by (F, v).

We can add to(V, v) a collar of $(\partial V, \partial v)$ in W to obtain a new handlebody $V\prime$ and a set of meridians $v\prime$ of $V\prime$. Similarly we create $(W\prime, w\prime)$, so that $V\prime \cap W'$ is a regular neighborhood $F \times [-1,1]$, where $F \times \{-1\} = \partial V'$ and $F \times \{1\} = \partial W'$. We also assume that $v\prime \cap w' = (\partial v \cap \partial w) \times [-1,1]$. In this way , if we assume that ∂v and ∂w cut transversely and that $\partial v \cup \partial w$ *fills* F (i.e. $\partial v \cup \partial w$ defines a cell decomposition of F), then $v\prime \cup w'$ defines a cell decomposition of $F \times [-1,1]$. It is now clear that the union of the Dehn-spheres $S(\partial V', v') \cup S(\partial W', w')$ separates M into a disjoint union of open 3-balls and defines a cell decomposition of M. In fact, $M - W'$ is separated into $g+1$ open 3-balls; (similarly $M - V'$); and $F \times [-1,1]$ is separated in as many open 3-balls as there are cells in the cell-decomposition C of F defined by $\partial v \cup \partial w$.Thus, each 2-cell e^2 of C gives rise to a product 3-cell $e^2 \ \times [-1,1]$; each 1-cell e^1 of C in $v - \partial v \cap \partial w$ (resp. in $w - \partial v \cap \partial w$) gives rise to a wedge-shaped 3-cell with the finer end in $e^1 \ \times \{-1\}$ (resp. in $e^1 \ \times \{1\}$); each 0-cell e^0 of C (in $\partial v \cap \partial w$) gives rise to a tetrahedron 3-cell $T(x)$ which is the intersection of two wedges (see Figures 1a, 1b, 1c, 1d).

There only remains to piping up $S(\partial V', v')$ and $S(\partial W', w')$ to get a filling Dehn-sphere. This is done as follows (see Figures 1d and 1e). Take and edge $\gamma = (16)$ of $T(x)$ connecting $.F \times \{-1\}$ and $F \times \{1\}$.(There are four of these edges in $T(x)$, and all of them are part of double curves.) The end points 1 and 6 of γ are triple points. We can pipe (see $[RS]$, page 67) $S(\partial V', v')$and $S(\partial W', w')$ along γ to obtain a new filling Dehn-sphere. Inspection shows that this process decreases the number of triple points by two and connects four open 3-balls (in the vicinity of 1) with

four corresponding open 3-balls (in the vicinity of 6), decreasing in this way the total number of open 3-balls by four. This ends the proof of the Theorem.

A simple counting argument shows that the filling Dehn-sphere S given by the above Theorem, decomposes M into $4t$ balls where t is the number of points in $\partial v \cap \partial w$. The number of triple points of S is $4t - 2$. Thus the number of 1-cells is $12t - 6$. The number of 2-cells is (Euler characteristic argument) $12t - 4$.

Thus the Lens-space $L(p,q)$ has a Dehn-sphere decomposing it into $4p$ balls. In particular $S^3 = L(1,0)$ can be decomposed in 4 balls.

The proof of the above Theorem is constructive and it is interesting to apply it to the simplest possible cases. Since the above argument uses that the Heegaard diagram fills F, one needs to start with an F of genus at least $g = 1$. Performing the construction in the proof of the last Theorem one gets the diagram of Figure 2 from the Heegaard diagrams of genus one of $.RP^3 = L(1,0)$. (The reader should compare Figures 1 and 2 to understand the algorithm).

The picture in Figure 2 is what is called a Johansson diagram (see $[J]$, $[H1]$). Conditions for a Johansson diagram to be realizable are given in $[J]$. One easily finds conditions for the diagram to represent a filling Dehn-sphere. The interest of Theorem 1 is that Johansson diagrams are a suitable way for representing all closed, orientable 3-manifolds. A Johansson diagram satisfying these conditions might be called a *Johansson representation.* An algorithm producing a Johansson representations out of a Heegaard diagram has been explained above. A number of interesting problems present themselves in this context, among them, the question in the last paragraph of $[H1]$.

Figure 1 a), b), c)

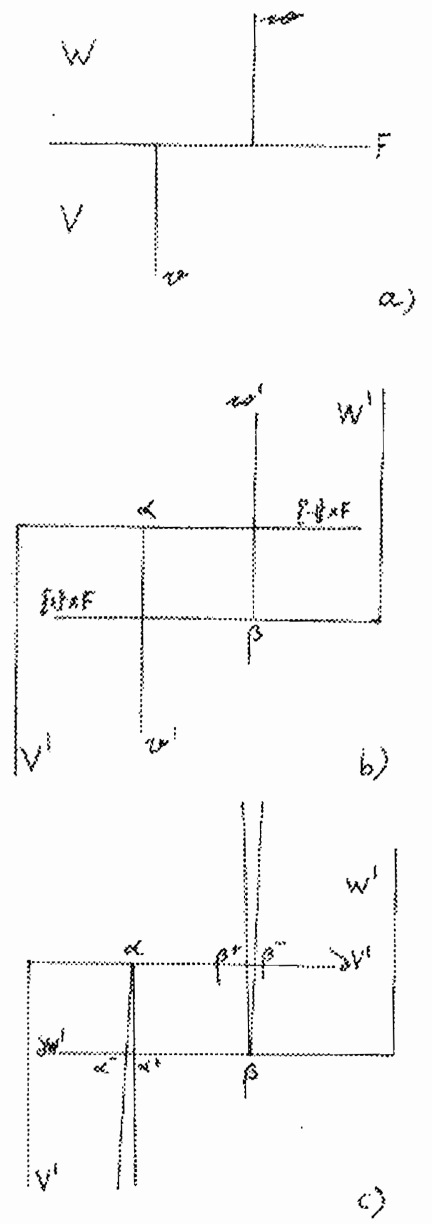


Figure 1

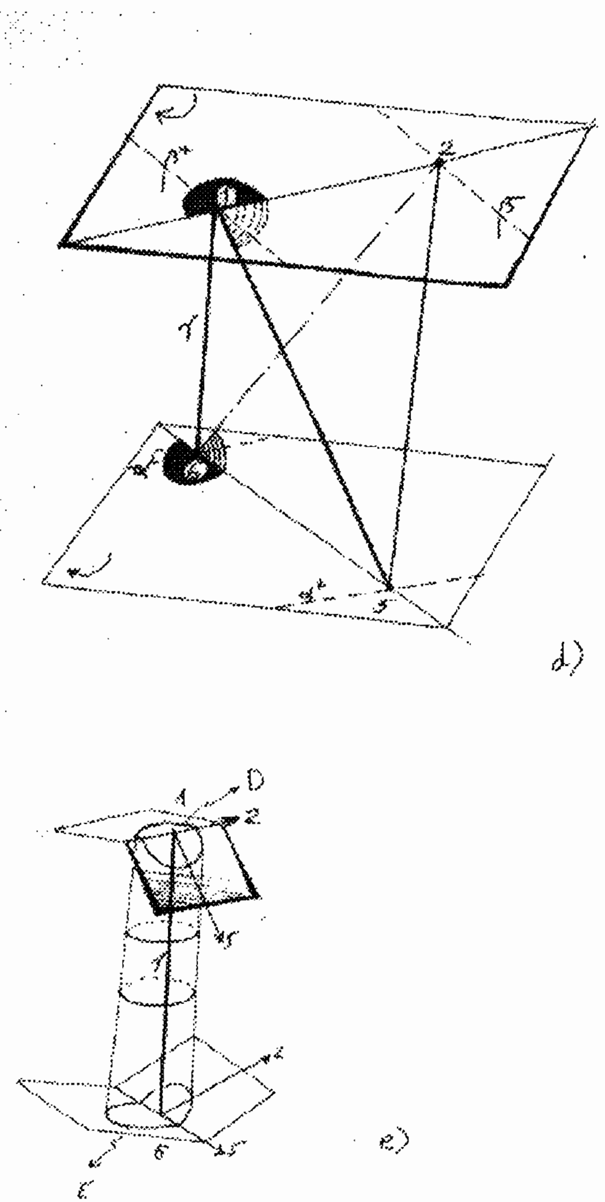


Figure 2a

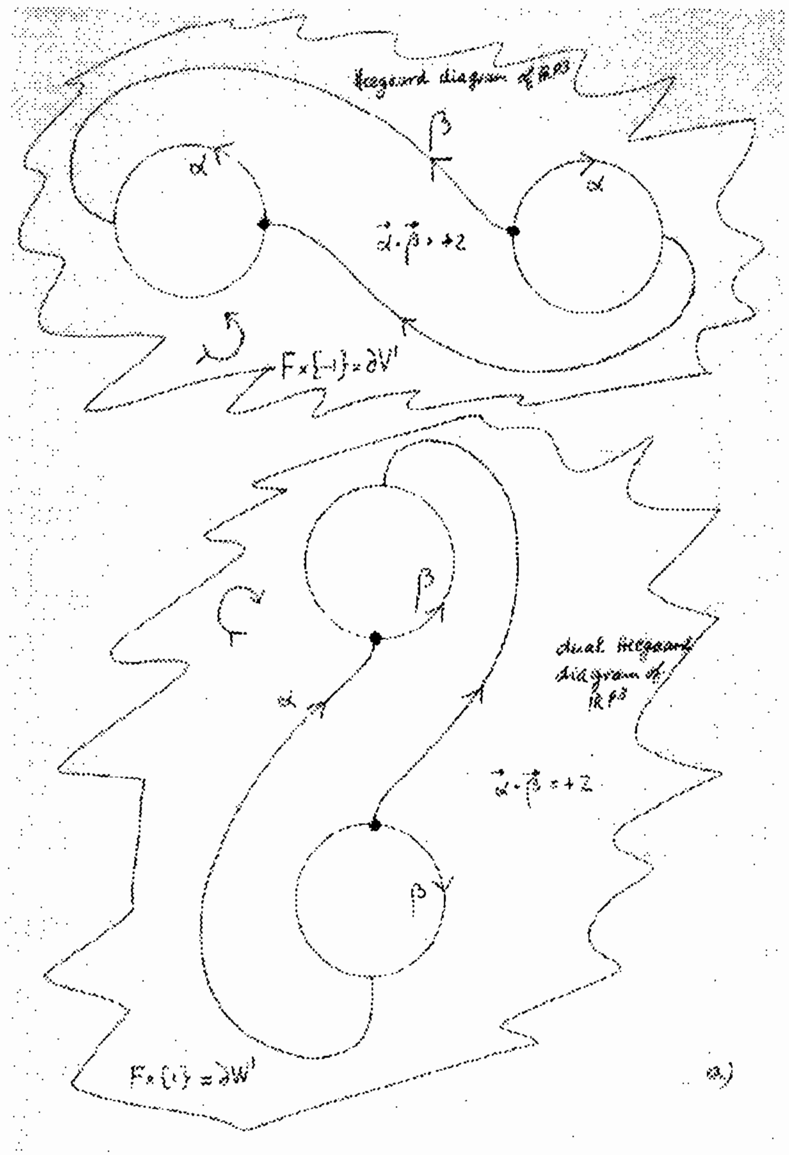


Figure 2b

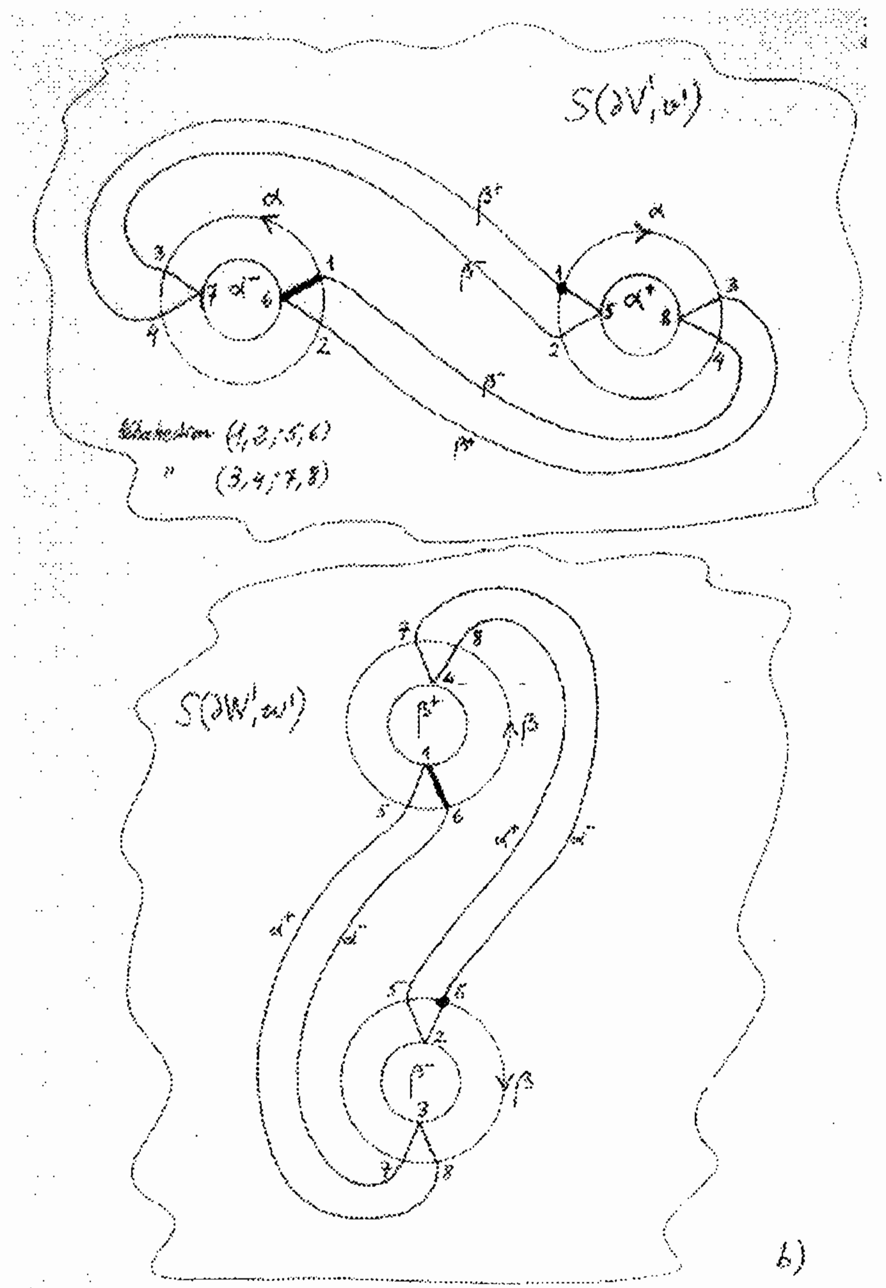


figure 2c

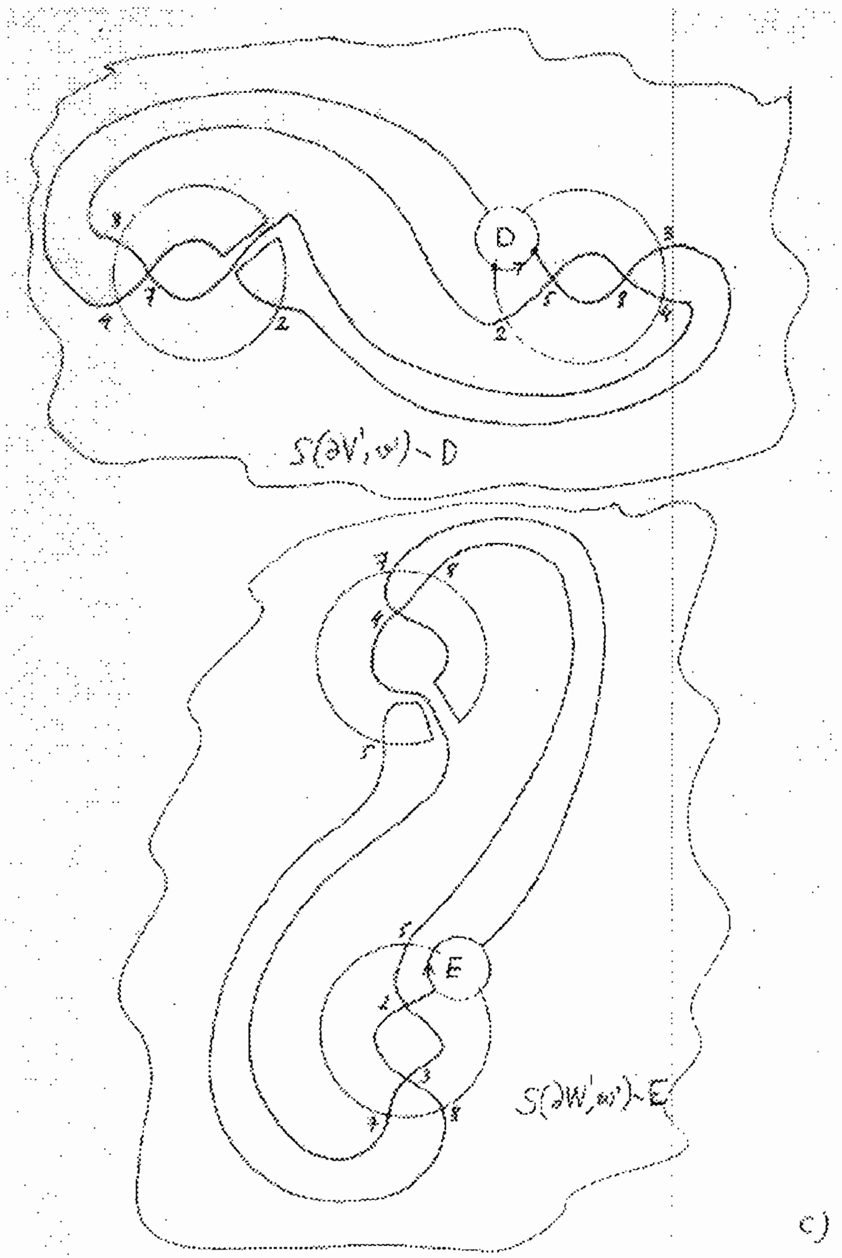
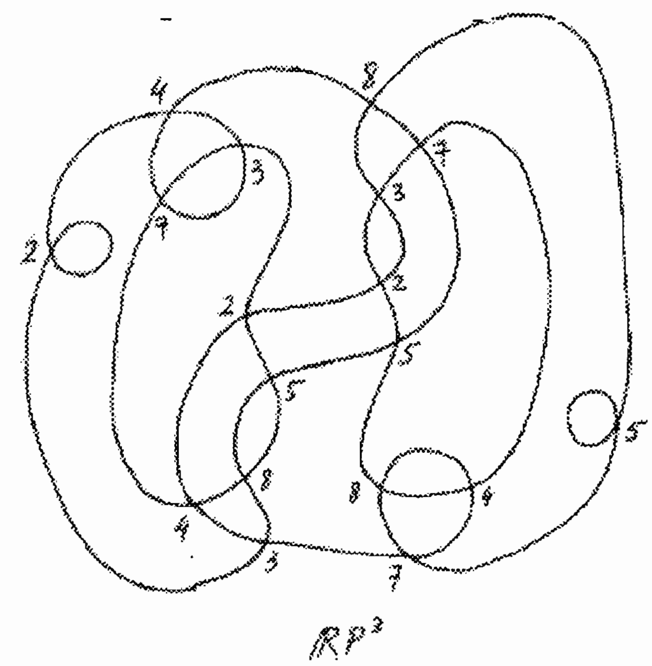


Figure 2d



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